

The Necessary Maximality Principle for c.c.c. forcing is equiconsistent with a weakly compact cardinal

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Abstract. The Necessary Maximality Principle for c.c.c. forcing with real parameters is equiconsistent with the existence of a weakly compact cardinal.

The Necessary Maximality Principle for c.c.c. forcing, denoted $\square\text{MP}_{\text{ccc}}(\mathbb{R})$, asserts that any statement about a real in a c.c.c. extension that could become true in a further c.c.c. extension and remain true in all subsequent c.c.c. extensions, is already true in the minimal extension containing the real. We show that this principle is equiconsistent with the existence of a weakly compact cardinal.

The principle is one of a family of principles considered in [Ham03] (building on ideas of [Cha00] and overlapping with independent work in [SV01]).

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The family begins with the Maximality Principle MP , the scheme asserting the truth of any statement that holds in some forcing extension $V^{\mathbb{P}}$ and all subsequent extensions $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ (these are the *forceably necessary* statements). The boldface form MP allows real parameters in the scheme, and the Necessary Maximality Principle $\Box\text{MP}$ asserts MP in all forcing extensions, using the parameters available in those extensions. The main results of [Ham03] show that MP is equiconsistent with ZFC, while MP is equiconsistent with the Lévy scheme “ORD is Mahlo” and $\Box\text{MP}$ is far stronger. Philip Welch proved that $\Box\text{MP}$ implies Projective Determinacy, and the second author of this paper improved the conclusion to $\text{AD}^{L(\mathbb{R})}$. He also provided an upper bound by proving the consistency of $\Box\text{MP}$ from the theory “ $\text{AD}_{\mathbb{R}} + \Theta$ is regular”.

In this article, we focus on the principles obtained by restricting attention to the class of c.c.c. forcing notions. The parameter-free version MP_{ccc} asserts the truth of any statement holding in some c.c.c. extension $V^{\mathbb{P}}$ and all subsequent c.c.c. extensions $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. This is equiconsistent with ZFC by [Ham03, Corollary 32]. An almost identical principle, where one requires the upward absoluteness of the statement from $V^{\mathbb{P}}$ to any $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ to be ZFC-provable (rather than merely true), was considered independently in [SV01].

The principle MP_{ccc} implies a spectacular failure of the Continuum Hypothesis. The reason is that with c.c.c. forcing one may add as many Cohen reals as desired, and once they are added, of course, the value of the continuum 2^{ω} remains inflated in all subsequent c.c.c. extensions. Thus, the assertion that 2^{ω} is larger than $\aleph_{\omega+1}$, say, or any cardinal whose definition is c.c.c. absolute, is c.c.c. forceably necessary, and hence true under MP_{ccc} .

For the boldface version of MP_{ccc} , there is initially little reason to restrict as in MP to real parameters, and so we denote by $\text{MP}_{\text{ccc}}(X)$ the scheme in which arbitrary parameters in X are allowed. Because for any parameter z the assertion $|z| < 2^{\omega}$ is c.c.c. forceably necessary, we can’t allow parameters outside $H(2^{\omega})$. Parameters inside $H(2^{\omega})$, however, are fine, and $\text{MP}_{\text{ccc}}(H(2^{\omega}))$ is equiconsistent with MP , which as we have mentioned is equiconsistent with the Lévy scheme (see [Ham03]). The weaker principle $\text{MP}_{\text{ccc}}(\mathbb{R})$ has recently been proved by Leibman [Lei04] to be equiconsistent with ZFC, and one may freely add a large initial segment of the ordinals as parameters.

The strongest form of the principle is $\Box\text{MP}_{\text{ccc}}(X)$, which asserts that $\text{MP}_{\text{ccc}}(X)$ holds in all c.c.c. extensions, reinterpreting X *de dicto* in these extensions. Thus, the principle asserts that if x is in X in some c.c.c. ex-

tension $V^{\mathbb{P}_0}$ and $\varphi(x)$ holds in a further c.c.c. extension $V^{\mathbb{P}_0 * \mathbb{P}}$ and all subsequent c.c.c. extensions $V^{\mathbb{P}_0 * \mathbb{P} * \dot{\mathbb{Q}}}$, then $\varphi(x)$ holds already in $V^{\mathbb{P}_0}$. Because $\text{MP}_{\text{ccc}}(H(2^\omega))$ is equiconsistent with MP , one might have expected the same for $\Box\text{MP}_{\text{ccc}}(H(2^\omega))$ and $\Box\text{MP}$. But the former principle is simply false.

Observation 1 (Leibman [Lei04]) $\Box\text{MP}_{\text{ccc}}(H(2^\omega))$ is false.

Leibman merely observed that $\text{MP}_{\text{ccc}}(H(2^\omega))$ implies Martin's Axiom MA, because the assertion that there is a filter for a given c.c.c. partial order meeting a certain family of dense sets is c.c.c. forceably necessary. Thus, if $\Box\text{MP}_{\text{ccc}}(H(2^\omega))$ held, then MA would hold in all c.c.c. extensions. But MA does not hold in all c.c.c. extensions, because even the forcing to add a single Cohen real creates Souslin trees. This argument makes an essential use of the uncountable parameters available in $H(2^\omega)$, such as the Souslin trees in the Cohen extension, and there appears to be no general way to get by with just real parameters (although doing so in the special case when ω_1 is accessible to reals is the key to Theorem 7 below).

So when it comes to the necessary form of the principle, the natural collection of parameters is \mathbb{R} after all, and we focus our attention on the principle $\Box\text{MP}_{\text{ccc}}(\mathbb{R})$.

Main Question 2 Is $\Box\text{MP}_{\text{ccc}}(\mathbb{R})$ consistent?

This question is answered by our main theorem.

Main Theorem 3 The principle $\Box\text{MP}_{\text{ccc}}(\mathbb{R})$ is equiconsistent over ZFC with the existence of a weakly compact cardinal.

The rest of this article consists of our proof of this theorem, followed by a short application of the proof to $\text{MP}(\mathbb{R})$. We concentrate first on the converse direction of the Main Theorem. Let $V_\delta \prec V$ denote the scheme, in the language with a constant symbol for δ , asserting for every formula φ in the language of set theory that $\forall x \in V_\delta [\varphi(x) \leftrightarrow \varphi(x)^{V_\delta}]$. The point of this is that in the construction of Theorem 5 we would like at heart to have a truth predicate for V , which is of course lacking by Tarski's theorem, but we can get by merely with a truth predicate for V_δ and the scheme $V_\delta \prec V$. Note that if $V_\delta \prec V$ and $G \subseteq \mathbb{P}$ is V -generic for forcing $\mathbb{P} \in V_\delta$, then $V_\delta[G] \prec V[G]$, because V_δ and V agree on whether a given statement is forced.

Lemma 4 *If there is a model of $\text{ZFC} + \text{there is a weakly compact cardinal}$, then there is a model of $\text{ZFC} + \text{there is a weakly compact cardinal} + V_\delta \prec V$.*

Proof: Let T be the latter theory, and suppose that M is a model of ZFC with a weakly compact cardinal. Since M satisfies every instance of the Lévy Reflection Theorem, it follows that every finite subset of T is consistent, by interpreting δ to be a sufficiently reflective ordinal of M . And so T as a whole is consistent. \square

The converse implication of the Main Theorem now follows from:

Theorem 5 *Assume κ is weakly compact, $\kappa < \delta$ and $V_\delta \prec V$. Then there is a forcing extension satisfying $\square_{\text{MP}_{\text{CCC}}}(\mathbb{R}) + \kappa = \omega_1$.*

The proof of this theorem relies in part on some general facts due to Kunen and Harrington-Shelah [HS85] concerning forcing and weakly compact cardinals. For completeness, we include proofs here.

Lemma 5.1 *If κ is weakly compact, then any finite support product of κ -c.c. forcing is κ -c.c.*

Proof: Suppose first that \mathbb{P} and \mathbb{Q} are κ -c.c. and that $A \subseteq \mathbb{P} \times \mathbb{Q}$ is an antichain of size κ in the product. Enumerate $A = \{(p_\alpha, q_\alpha) \mid \alpha < \kappa\}$ and define the coloring $f : [\kappa]^2 \rightarrow 2$ by $f(\alpha, \beta) = 0$ if $p_\alpha \perp p_\beta$, otherwise 1. Since κ is weakly compact, there is a homogeneous set $H \subseteq \kappa$ of size κ , meaning that f is constant on $[H]^2$. If the constant value is 0, then $p_\alpha \perp p_\beta$ for all $\alpha, \beta \in H$, contradicting that \mathbb{P} is κ -c.c. Otherwise the constant value is 1, in which case $q_\alpha \perp q_\beta$ for all such α and β , contradicting that \mathbb{Q} is κ -c.c. By induction, it follows that any finite product of κ -c.c. forcing is κ -c.c. Consider now an antichain A of size κ in an arbitrary finite-support product $\prod_{\alpha \in I} \mathbb{P}_\alpha$, where each \mathbb{P}_α is κ -c.c. By a delta system argument, we may assume that supports of the conditions in A form a delta system. Any two conditions in A must be incompatible on the root of this system, contradicting the fact that any finite product of κ -c.c. partial orders is κ -c.c. \square

Lemma 5.2 *If κ is weakly compact and \mathbb{B} is a κ -c.c. complete Boolean algebra, then every subset $A \subseteq \mathbb{B}$ of size less than κ generates a complete subalgebra that is also of size less than κ .*

Proof: Construct the increasing continuous sequence of subalgebras $A = A_0 \subseteq A_1 \subseteq \dots \subseteq A_\alpha \subseteq \dots$, for $\alpha < \kappa$, where $A_{\alpha+1}$ is obtained by adding to A_α the infima (computed in \mathbb{B}) of all subsets of A_α and closing under the Boolean operations, taking unions at limits. Let $\mathbb{A} = \bigcup_\alpha A_\alpha$. Since \mathbb{B} is κ -c.c., all the antichains of \mathbb{A} live on some A_α , and so \mathbb{A} is the complete subalgebra generated by A in \mathbb{B} . Since $|A_\alpha| < \kappa$ for all $\alpha < \kappa$, it follows that $|\mathbb{A}| \leq \kappa$. By moving to an isomorphic copy of \mathbb{B} , assume $\mathbb{A} \subseteq \kappa$ and place \mathbb{A} into a transitive model of set theory M of size κ . Fix a weakly compact embedding $j : M \rightarrow N$ with critical point κ , and observe in N that \mathbb{A} is a complete subalgebra of $j(\mathbb{A})$ containing the generators A . Thus, $\mathbb{A} = j(\mathbb{A})$ and so $|\mathbb{A}| < \kappa$, as desired. \square

Lemma 5.3 *Consequently, if κ is weakly compact and $G \subseteq \mathbb{P}$ is V -generic for κ -c.c. forcing \mathbb{P} , then every $x \in H(\kappa)^{V[G]}$ is V -generic for κ -c.c. forcing of size less than κ .*

Proof: Let \mathbb{B} be the regular open algebra of \mathbb{P} . By coding, we may assume that $x \subseteq \beta$ for some $\beta < \kappa$. Let \dot{x} be a \mathbb{P} -name for x such that $\Vdash_{\mathbb{P}} \dot{x} \subseteq \check{\beta}$. For each $\xi < \beta$, let $b_\xi = \llbracket \xi \in \dot{x} \rrbracket$. By the previous lemma, the complete subalgebra \mathbb{A} generated by $A = \{b_\xi \mid \xi < \beta\}$ has size less than κ . And clearly x is constructible from $G \cap A$. \square

Our proof also relies on the term forcing construction, which we now explain. Suppose that \mathbb{P} is any partial order and $\dot{\mathbb{Q}}$ is the \mathbb{P} -name of a partial order. The term forcing \mathbb{Q}_{term} for $\dot{\mathbb{Q}}$ over \mathbb{P} consists of conditions q such that $\mathbb{1} \Vdash q \in \dot{\mathbb{Q}}$, with the order $p \leq_{\text{term}} q$ if and only if $\mathbb{1} \Vdash p \leq_{\dot{\mathbb{Q}}} q$. One can restrict the size of \mathbb{Q}_{term} by using only the names p in a full set B of names, meaning that for any \mathbb{P} -name q with $\mathbb{1} \Vdash q \in \dot{\mathbb{Q}}$ there is $p \in B$ with $\mathbb{1} \Vdash p = q$. Any such full set of names forms a dense subset of \mathbb{Q}_{term} , which is therefore equivalent as a forcing notion. The fundamental property of term forcing is the following:

Lemma 5.4 *Suppose that $H_{\text{term}} \subseteq \mathbb{Q}_{\text{term}}$ is V -generic for the term forcing of $\dot{\mathbb{Q}}$ over \mathbb{P} and \overline{V} is any model of set theory with $V[H_{\text{term}}] \subseteq \overline{V}$. If there is a V -generic filter $G \subseteq \mathbb{P}$ in \overline{V} , then there is a $V[G]$ -generic filter $H \subseteq \mathbb{Q} = \dot{\mathbb{Q}}_G$ in \overline{V} .*

Proof: The filter H_{term} consists of \mathbb{P} -names for conditions in $\dot{\mathbb{Q}}$, so it makes sense to let $H = \{q_G \mid q \in H_{\text{term}}\}$ in \bar{V} . To see that this is $V[G]$ -generic for $\mathbb{Q} = \dot{\mathbb{Q}}_G$, suppose that $D \subseteq \mathbb{Q}$ is a dense subset of \mathbb{Q} in $V[G]$. Let \dot{D} be a \mathbb{P} -name for D , forced by $\mathbb{1}$ to be dense, and let D_{term} be the set of conditions \dot{q} that are forced by $\mathbb{1}$ to be in \dot{D} . It is easy to see that D_{term} is a dense subset of \mathbb{Q}_{term} , and so there is a condition $q \in H_{\text{term}} \cap D_{\text{term}}$. It follows that $q_G \in H \cap D$, and so H meets D , as desired. \square

In particular, if $\bar{V} = V[H_{\text{term}}][B]$ is a forcing extension of $V[H_{\text{term}}]$ containing a V -generic $G \subseteq \mathbb{P}$, then we may rearrange the forcing as $\bar{V} = V[H_{\text{term}}][B] = V[G][H][(H_{\text{term}} * B)/(G * H)]$, where $(H_{\text{term}} * B)/(G * H)$ is the quotient forcing adding the rest of $H_{\text{term}} * B$ over $V[G][H]$. There is no need in Lemma 5.4 for G and H_{term} to be mutually V -generic.

Lemma 5.5 *In the context of the previous Lemma, if κ is weakly compact in V , $|\mathbb{P}| < \kappa$ and $\mathbb{1} \Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\check{\kappa}$ -c.c., then \mathbb{Q}_{term} is κ -c.c. in V (and hence also in $V[G]$). If $\bar{V} = V[H_{\text{term}}][B]$ is a κ -c.c. forcing extension of $V[H_{\text{term}}]$, then it is a κ -c.c. extension of the resulting $V[G][H]$.*

Proof: Suppose that $A \subseteq \mathbb{Q}_{\text{term}}$ is an antichain in V of size κ . Any two elements $q_0, q_1 \in A$ are incompatible in \mathbb{Q}_{term} , meaning that there is no condition $q \in \mathbb{Q}_{\text{term}}$ such that $\mathbb{1}_{\mathbb{P}}$ forces both $q \leq_{\dot{\mathbb{Q}}} q_0$ and $q \leq_{\dot{\mathbb{Q}}} q_1$. It follows that there is some condition $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} q_0 \perp_{\dot{\mathbb{Q}}} q_1$. Enumerate $A = \{q_\beta \mid \beta < \kappa\}$ and define $f : [\kappa]^2 \rightarrow \mathbb{P}$ by $f(\alpha, \beta) = p$ for some p forcing $q_\alpha \perp_{\dot{\mathbb{Q}}} q_\beta$. Since κ is weakly compact, there is a homogeneous set $H \subseteq \kappa$ of size κ on which f has some constant value p . Thus, $p \Vdash q_\alpha \perp_{\dot{\mathbb{Q}}} q_\beta$ for all $\alpha < \beta$ from H , contradicting that $\mathbb{1}$ forces $\dot{\mathbb{Q}}$ is $\check{\kappa}$ -c.c. So \mathbb{Q}_{term} is κ -c.c. in V .

If $\bar{V} = V[H_{\text{term}}][B]$ is a forcing extension of $V[H_{\text{term}}]$, then we have already observed that $\bar{V} = V[G][H][(H_{\text{term}} * B)/(G * H)]$ is obtained by quotient forcing over $V[G][H]$. Since $H_{\text{term}} * B$ is κ -c.c., the proof is completed by the fact that any quotient of κ -c.c. forcing is κ -c.c. \square

Putting all this together, we now prove Theorem 5.

Proof of Theorem 5: We assume $V_\delta \prec V$ and κ is a weakly compact cardinal below δ . Let \mathcal{I} be the set of pairs $\langle \mathbb{P} * \dot{\mathbb{Q}}, \varphi(\dot{x}) \rangle$ in V_δ such that $\mathbb{P} \in V_\kappa$, \dot{x} is a \mathbb{P} -name for an element of $H(\kappa)$ and $\dot{\mathbb{Q}} \in V_\delta^\mathbb{P}$ is further κ -c.c. forcing such that $\mathbb{1}$ forces via $\mathbb{P} * \dot{\mathbb{Q}}$ over V_δ that $\varphi(\dot{x})$ is true in all κ -c.c. forcing extensions of $V_\delta^{\mathbb{P} * \dot{\mathbb{Q}}}$. In this case, let $\mathbb{Q}_{(\mathbb{P} * \dot{\mathbb{Q}}, \varphi(\dot{x}))}$ be the term-forcing for $\dot{\mathbb{Q}}$ over \mathbb{P} , and let

$\mathbb{Q}_\infty = \Pi_{\mathcal{I}} \mathbb{Q}_{\langle \mathbb{P} * \dot{\mathbb{Q}}, \varphi(\dot{x}) \rangle}$ be the finite support product of these posets. By Lemma 5.5, each factor in this poset is κ -c.c., and so by Lemma 5.1 the product \mathbb{Q}_∞ is also κ -c.c. Suppose that $G_\infty \subseteq \mathbb{Q}_\infty$ is V -generic and consider $V[G_\infty]$. Suppose that \mathbb{P} is some forcing in $V[G_\infty]$ adding an object \dot{x} in $H(\kappa)$, and that $\varphi(\dot{x})$ is forceably necessary for κ -c.c. forcing over $V[G_\infty]^\mathbb{P}$. That is, there is some further κ -c.c. forcing $\dot{\mathbb{Q}}$ such that $\varphi(\dot{x})$ holds in all κ -c.c. extensions of $V[G_\infty]^\mathbb{P} * \dot{\mathbb{Q}}$. Since $\mathbb{Q}_\infty * \mathbb{P}$ adds \dot{x} , there is by Lemma 5.3 a complete suborder $\mathbb{P}_0 \subseteq \mathbb{Q}_\infty * \mathbb{P}$ of size less than κ adding \dot{x} , so let us assume that \dot{x} is a \mathbb{P}_0 -name. Since $V[G_\infty]^\mathbb{P} * \dot{\mathbb{Q}}$ is a κ -c.c. extension of $V^{\mathbb{P}_0}$, it follows that $\varphi(\dot{x})$ is κ -c.c. forceably necessary over $V^{\mathbb{P}_0}$, and therefore, by the elementarity $V_\delta^{\mathbb{P}_0} \prec V^{\mathbb{P}_0}$, it is κ -c.c. forceably necessary over $V_\delta^{\mathbb{P}_0}$. So we may assume that $\dot{\mathbb{Q}}$ was chosen from $V_\delta^{\mathbb{P}_0}$, and that it is forced by $\mathbb{P}_0 * \dot{\mathbb{Q}}$ that $\varphi(\dot{x})$ holds in all κ -c.c. extensions of $V_\delta^{\mathbb{P}_0} * \dot{\mathbb{Q}}$. Thus, $\langle \mathbb{P}_0 * \dot{\mathbb{Q}}, \varphi(\dot{x}) \rangle \in \mathcal{I}$, and $V[G_\infty]$ has a V -generic filter H_{term} for the term forcing $\mathbb{Q}_{\langle \mathbb{P}_0 * \dot{\mathbb{Q}}, \varphi(\dot{x}) \rangle}$. Since $G_0 \in V[G_\infty]$, it follows by Lemma 5.4 that there is a $V[G_0]$ -generic filter $H \subseteq \mathbb{Q} = \dot{\mathbb{Q}}_{G_0}$ in $V[G_\infty]$. By the choice of $\dot{\mathbb{Q}}$, we know that $\varphi(x)$ holds in all κ -c.c. extensions of $V_\delta[G_0][H]$, and hence by elementarity in all κ -c.c. extensions of $V[G_0][H]$. Since $V[G_\infty]$ is by Lemma 5.5 a κ -c.c. extension of $V[G_0][H]$, we conclude that $\varphi(x)$ holds there, as desired. We have established that $V[G_\infty]$ satisfies $\square \text{MP}_{\kappa\text{-c.c.}}(H(\kappa))$. It follows that κ has become ω_1 in $V[G_\infty]$, because for any $\alpha < \kappa$ the assertion that α is countable is κ -c.c. forceably necessary, and hence true in $V[G_\infty]$. So κ -c.c. has become the same as c.c.c., and we have $V[G_\infty] \models \square \text{MP}_{\text{ccc}}(\mathbb{R})$. \square

One may not omit the hypothesis of $V_\delta \prec V$ in Theorem 5, because if there is a model of ZFC with a weakly compact cardinal, then there is such a model having no forcing extension that is a model of MP_{ccc} . To see this, following [Ham03, Corollary 32], suppose M is any model of ZFC with a weakly compact cardinal. Let N be the union of all L_θ in this model, where θ is definable in L^M without parameters. An easy Tarski-Vaught argument shows that $N \prec L^M$, and so $N \models \text{ZFC} + V = L$ and the definable ordinals of N are unbounded in N . Note that the least weakly compact cardinal of L^M must be in N and weakly compact there. If θ is defined by $\varphi(x)$ in N , then $\varphi(x)^L$ defines θ in any forcing extension of N . Consequently, if such an extension $N[G]$ satisfies MP_{ccc} , then it would have to satisfy $2^\omega > \theta$, since this is expressible without parameters and is c.c.c. forceably necessary. Since the definable ordinals θ are unbounded in the ordinals of N , the value of 2^ω

in $N[G]$ would have to be larger than every ordinal, a contradiction.

This completes the converse direction of the Main Theorem 3. Let us turn now to the forward implication by showing that if the Necessary Maximality Principle $\square\text{MP}_{\text{ccc}}(\mathbb{R})$ is consistent with ZFC, then so is the existence of a weakly compact cardinal. This argument proceeds to a great measure merely by placing the known results of Theorems 6 and 7 adjacent to one another and observing the result.

Theorem 6 (Harrington-Shelah [HS85, Theorem C.i]) *If MA holds and ω_1 is inaccessible to reals, then it is weakly compact in L .*

Theorem 7 (Leibman [Lei04]) *If $\square\text{MP}_{\text{ccc}}(\mathbb{R})$ holds, then ω_1 is inaccessible to reals.*

The former result has been widely discussed elsewhere (for example, see [Sch01]). Leibman's proof of Theorem 7 proceeds roughly as follows: If $c : \omega \rightarrow \omega$ is a Cohen real added over V , then by Todorćević's proof of Shelah's theorem (see [Jec03, Theorem 28.12]) there is in $V[c]$ a Souslin tree $T(c)$ constructed by composing c with each element of an almost coherent family of injective functions $e_\alpha : \alpha \rightarrow \omega$. If $\omega_1 = \omega_1^{L[z]}$ for some real z in V , then there will be such an almost coherent family of injective function in $L[z]$. By using the $L[z]$ -least such family, the tree $T(c)$ is seen to be definable in $V[c]$ from the parameters z and c , and furthermore, this definition is absolute to any c.c.c. extension. The assertion that this tree has an ω_1 branch, therefore, which uses only the parameters z and c , is c.c.c. forceably necessary (since one can force with the Souslin tree), but not true in $V[c]$ (since it is a Souslin tree there), contradicting $\square\text{MP}_{\text{ccc}}(\mathbb{R})$.

Theorems 6 and 7 now combine to establish the result we need:

Corollary 8 *If $\square\text{MP}_{\text{ccc}}(\mathbb{R})$ holds, then ω_1 is weakly compact in L .*

Proof: If $\square\text{MP}_{\text{ccc}}(\mathbb{R})$ holds, then of course it holds in every c.c.c. extension. Consequently, by Theorem 7, if $\square\text{MP}_{\text{ccc}}(\mathbb{R})$ holds, then ω_1 is inaccessible to reals in every c.c.c. extension. Since there is such an extension satisfying MA, it follows by applying Theorem 6 in that extension that ω_1 is weakly compact in L , as desired. \square

Let us give a second proof, along a different route. The first step is an easy induction on formulas:

Theorem 9 *If $\Box\text{MP}_{\text{CCC}}(\mathbb{R})$ holds, then projective assertions are absolute by c.c.c. forcing.*

Proof: The method of [Ham03, Theorem 19] works generally. First notice that if $\Box\text{MP}_{\text{CCC}}(\mathbb{R})$ holds, then it holds in all c.c.c. extensions. Suppose inductively that a projective assertion $\varphi(x, y)$ is absolute from any c.c.c. extension to any further c.c.c. extension. Boolean combinations are easily preserved as well, so it suffices to consider the existential case. If $\exists x \varphi(x, a)$ is true in $V^{\mathbb{P}}$, then by substituting the witness into place, this is preserved by induction to any further extension $V^{\mathbb{P} * \dot{\mathbb{Q}}}$. Conversely, if $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models \exists x \varphi(x, a)$, then the existence of a witness is forceably necessary over $V^{\mathbb{P}}$, and hence true in $V^{\mathbb{P}}$, as desired. \square

Since c.c.c. projective absoluteness is known to be equiconsistent with the existence of a weakly compact cardinal, Corollary 7 now follows. [BF01] shows already that c.c.c. Σ_4^1 absoluteness is equiconsistent with a weakly compact cardinal.

Let us close the paper with an application of our method to the case of the original Maximality Principle MP , without restricting to c.c.c. forcing. By [Ham03, Theorem 12] we know that it is consistent that $\text{MP}(\mathbb{R})$ is indestructible by the forcing to add Cohen reals, but in fact $\text{MP}(\mathbb{R})$ is always indestructible.

Theorem 10 *The boldface Maximality Principle $\text{MP}(\mathbb{R})$, if true, is indestructible by the forcing to add any number of Cohen reals.*

Proof: Suppose first that $\text{MP}(\mathbb{R})$ holds in V and $V[c]$ is a generic extension obtained by adding a Cohen real c . To show that $V[c]$ models $\text{MP}(\mathbb{R})$, suppose that x is a real in $V[c]$ and $\varphi(x)$ is forceably necessary in $V[c]$. Thus, there is a poset \mathbb{Q} in $V[c]$ such that if $G \subseteq \mathbb{Q}$ is $V[c]$ -generic, then $\varphi(x)$ holds in $V[c][G]$ and all extensions. Let \dot{x} be a name for x and $\dot{\mathbb{Q}}$ a name for \mathbb{Q} , such that there is a condition p_0 in \mathbb{C} forcing that $\dot{\mathbb{Q}}$ makes $\varphi(\dot{x})$ necessary. Let \mathbb{Q}_{term} be the term forcing poset of $\dot{\mathbb{Q}}$ over the Cohen real forcing \mathbb{C} , and suppose $G_{\text{term}} \subseteq \mathbb{Q}_{\text{term}}$ is $V[c]$ -generic. The model $V[c][G_{\text{term}}] = V[G_{\text{term}}][c]$ has a V -generic $c \subseteq \mathbb{C}$ and $G_{\text{term}} \subseteq \mathbb{Q}_{\text{term}}$, so by the fundamental property of term forcing Lemma 5.4, there is a $V[c]$ -generic filter $G \subseteq \mathbb{Q}$ in $V[c][G_{\text{term}}]$ and we may view the extension as $V[G_{\text{term}}][c] = V[c][G][G_{\text{term}}/G]$, that is, as an extension of $V[c][G]$. Since $\varphi(x)$ was made necessary in $V[c][G]$, it follows that $V[G_{\text{term}}][c] \models \varphi(x)$. In fact we know that $p_0 \Vdash_{\mathbb{C}} \varphi(\dot{x})$ in $V[G_{\text{term}}]$.

Furthermore, if $V[G_{\text{term}}][H]$ is some other forcing extension of $V[G_{\text{term}}]$, and c is any $V[G_{\text{term}}][H]$ generic below p_0 , then we may rearrange the resulting extension as $V[G_{\text{term}}][H][c] = V[G_{\text{term}}][c][H] = V[c][G][G_{\text{term}}/G][H]$, which still satisfies $\varphi(x)$ since it is an extension of $V[c][G]$. Therefore, $p_0 \Vdash_{\mathbb{C}} \varphi(\dot{x})$ in $V[G_{\text{term}}][H]$ as well. In short, we have proved that the assertion “ $p_0 \Vdash_{\mathbb{C}} \varphi(\dot{x})$ ” is necessary in $V[G_{\text{term}}]$, and therefore forceably necessary in V . Since the parameters \mathbb{C} , p_0 and \dot{x} in this assertion are all hereditarily countable in V , we conclude by $\text{MP}(\mathbb{R})$ in V that the assertion must be true in V . Thus, since p_0 is in c , we conclude that $V[c] \models \varphi(x)$, without any need for the term forcing, as desired.

Now we prove the full result of the theorem. Suppose that $G \subseteq \text{add}(\omega, \kappa)$ is V -generic for the forcing to add κ many Cohen reals. By the countable chain condition, every real x of $V[G]$ is in $V[G \cap a]$ for some countable set $a \in V$. Since $G \cap a$ is isomorphic to adding a single Cohen real, we know by the previous paragraph that $\text{MP}(\mathbb{R}^{V[G \cap a]})$ is true in $V[G \cap a]$. In particular, any instance of the MP scheme involving the parameter x holds in $V[G \cap a]$. Since $V[G]$ is a forcing extension of $V[G \cap a]$, it follows that if $\varphi(x)$ is forceably necessary over $V[G]$, then it is forceably necessary over $V[G \cap a]$, and hence necessary in $V[G \cap a]$, and hence still necessary in $V[G]$. So we have established every instance of the $\text{MP}(\mathbb{R})$ scheme in $V[G]$, as desired. \square

The theorem applies more generally, of course, to any forcing extension all of whose reals are captured by Cohen forcing. We note the contrast of Theorem 10 with the consequence of Leibman’s proof of Observation 1, that $\text{MP}_{\text{CCC}}(H(2^\omega))$ is always destroyed by the forcing to add a Cohen real. It follows that if $\square \text{MP}_{\text{CCC}}(\mathbb{R})$ is consistent, it is consistent with the failure of $\text{MP}_{\text{CCC}}(H(2^\omega))$.

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